Differentiability

In one variable

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

the derivative at one point $a \in \mathbb{R}$ provides us with ratio of change of the function f at the point $x=a$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a) = \frac{df(a)}{dx}$$

Geometrically, we have a tangent line at $x=a$

$$\boxed{y = f(a) + f'(a)(x-a)}$$

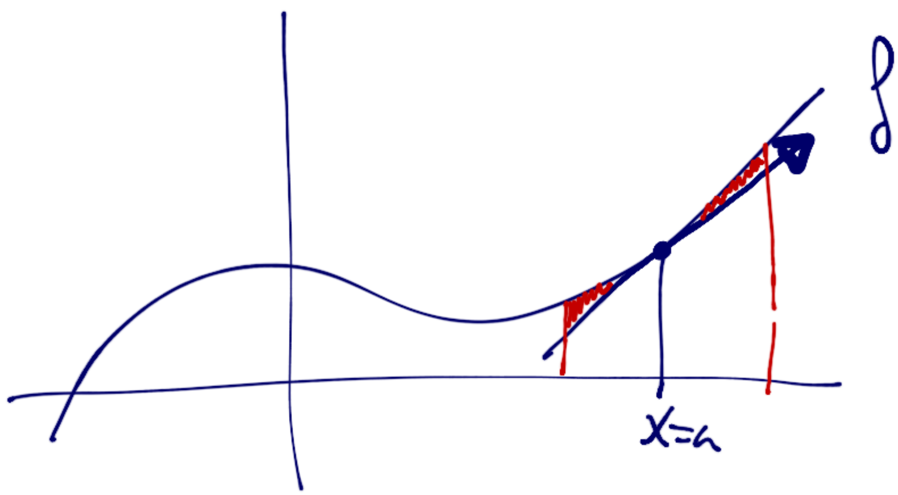
If we are very close to $(a, f(a))$

$$\boxed{f(x) = \overbrace{f(a) + f'(a)(x-a)}^{\text{tangent line}} + r(h)}$$

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{approximation}}$ $h \sim 0$ (Taylor's polyno)

$h = |x-a|$

$r(h) \equiv$ distance between the curve $f(x)$ and the tangent line.



For differentiability we need something else.

In fact, we need that

$$\lim_{h \to 0} \frac{r(h)}{h} = 0$$

This guarantees the diff.

Indeed,

$$f(x) - f(a) = f'(a)(x-a) + r(|x-a|)$$

$$\Downarrow$$

$$\frac{f(x)-f(a)}{x-a} = f'(a) + \frac{r(|x-a|)}{x-a}$$

If $x = a + h$

$f'(a)$ $f(a)$ 0

In one variable

differentiability $\equiv$ existence of derivatives.

We want to generalise these ideas to several variables.

For example

$$z = f(x,y) \ , \quad f: \mathbb{R}^2 \to \mathbb{R}.$$

First, we need to understand the ratio of change for each variable. (derivative)

- We perform a derivative of the function with respect to each variable.

Definition

Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a function and $x_0 \in A$

Then, we define the partial derivative of f with respect to the variable x_i as

$$\frac{\partial f(x_0)}{\partial x_i} = \lim_{t \to 0} \frac{f(x_0 + t e_i) - f(x_0)}{t}$$

$$= \lim_{t \to 0} \frac{f(x_1, \dots, x_i + t, \dots x_n) - f(x_1, \dots x_n)}{t}$$

$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \leftarrow$ position i , $e_i \equiv$ vectors of the

Example: • $f(x,y) = xy + x - y$ at $(0,0)$, $\dfrac{\partial f(0,0)}{\partial x}$ $\dfrac{\partial f(0,0)}{\partial y}$?

$$\frac{\partial f(0,0)}{\partial x} = \lim_{h \to 0} \frac{f(0+h,0) - \overbrace{f(0,0)}^{=0}}{h} = \lim_{h\to 0} \frac{\overbrace{h\cdot 0}^{=0} + h - 0 - \overbrace{0}^{=0}}{h}$$

$$= 1$$

$$\frac{\partial f(0,0)}{\partial y} = \lim_{k\to 0} \frac{f(0,\overbrace{k}^{0+k}) - f(0,0)}{k} = \lim_{k \to 0} \frac{-k}{k} = -1$$

• $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ $\leftarrow f(0,0)=0$

$$\frac{\partial f(0,0)}{\partial x} = \lim_{h\to 0} \frac{f(0+h,0) - \overbrace{f(0,0)}^{=0}}{h} = \lim \frac{\frac{0}{h^2+0}}{h} = 0$$

∂y $k \to 0$ $k \to 0$ k

$$f(x,y) = xy + x - y$$

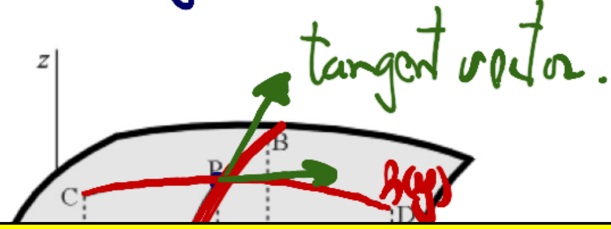
$$\underbrace{\frac{\partial f(x,y)}{\partial x} = y+1}_{\Downarrow} \quad , \quad \underbrace{\frac{\partial f(x,y)}{\partial y} = x-1}_{\Downarrow}$$

$$\frac{\partial f(0,0)}{\partial x} = 1 \qquad\qquad \frac{\partial f(0,0)}{\partial y} = -1$$

Geometric interpretation

Assume $z = f(x,y)$ and we fix one of the variables.

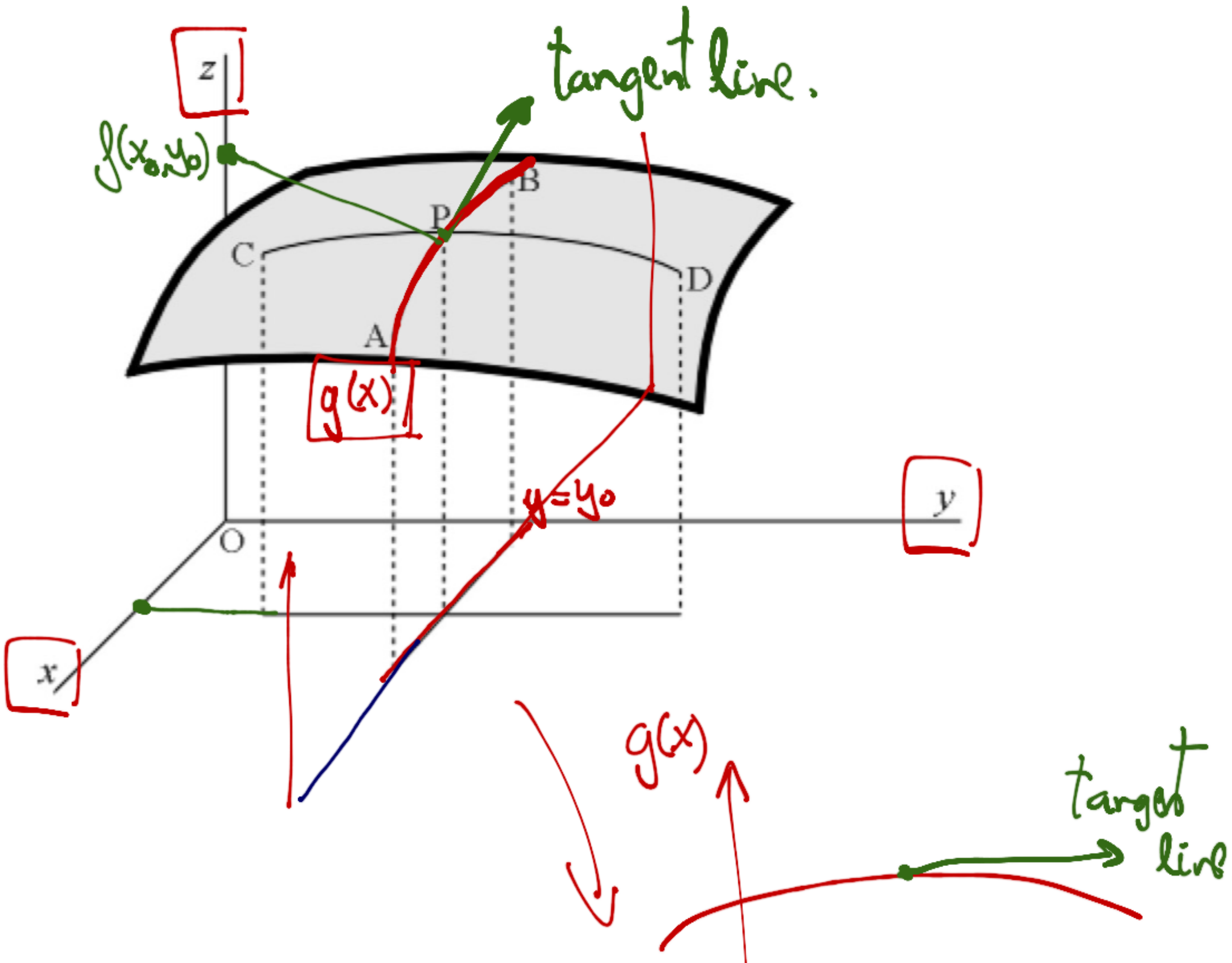
Vertical plane $\left.\begin{array}{l} y = y_0 \\ z = f(x,y) \end{array}\right\} \Rightarrow z = f(x,y_0) = g(x)$



Also,

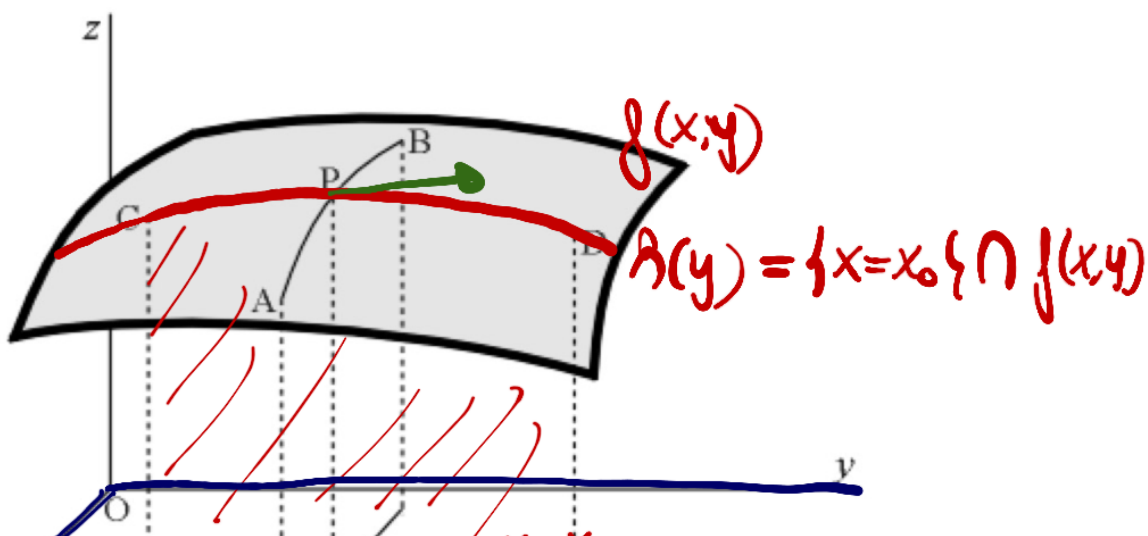
$y = y_0$

The partial derivative of f with respect to x at P will be represented by the tangent line to the curve $g(x) = f(x, y_0)$



Similarly, the partial derivative of f with respect to y at P will be the tangent line at to the graph of $h(y)$

$$\left.\begin{array}{l} z = f(x,y) \\ x = x_0 \end{array}\right\} \Rightarrow z = f(x_0, y) = h(y)$$



in the directions of x and y.

We can extend those ideas to any direction.

Definition – Directional derivatives

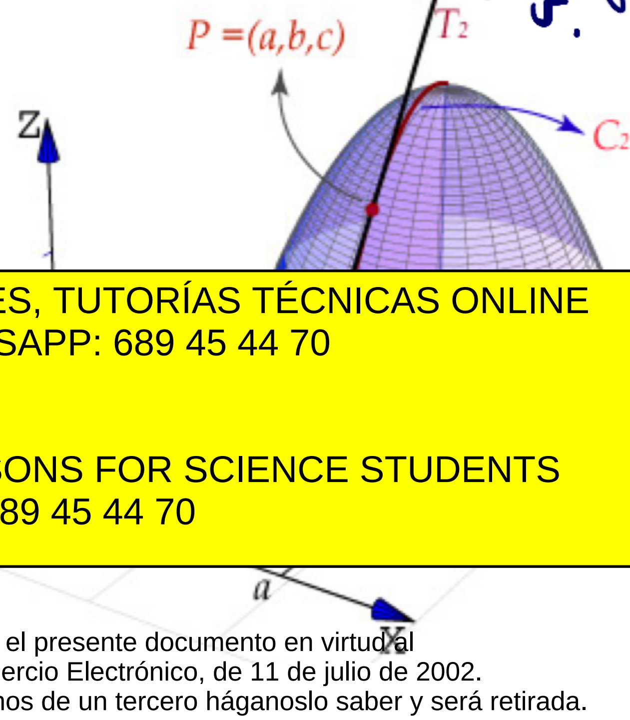
$f: A \subset \mathbb{R}^n \to \mathbb{R}$, $P \in A \subset \mathbb{R}^n$ a point in dom f.

and $v \in \mathbb{R}^n \setminus \{0\}$ any vector.

Then, we say the f has derivative at P on the direction of v when the following limit exists.

$$\lim_{t \to 0} \frac{f(P+tv) - f(P)}{t\|v\|}$$

derivative following v.



Remarks

- The concept of directional derivative generalises the concept of partial derivatives to any direction

$$D_v f(P) = \lim_{t \to 0} \frac{f(P+tv) - f(P)}{t\|v\|} = \frac{d}{dt} f(P+tv)\Big|_{t=0}$$

$f : \mathbb{R}^N \to \mathbb{R}$.

$f(P) \in \mathbb{R}$, $P \in \mathbb{R}^N$

How f changes $\rightarrow$ $\dfrac{f(P+tv) - f(P)}{\underbrace{\|P+tv-P\|}_{=\|tv\| = t\|v\|}}$ $= dist(P, P+tv)$

in relation to P and $P+tv$ $\rightarrow$

In 1D

$$\frac{f(x+t) - f(x)}{\;} = \text{ratio}$$

Examples: $f(x,y)=\sqrt{|xy|}$

Compute the derivative at $(0,0)$ in the direction of $v=(1,1)$

$$D_v f(0,0)=\lim_{t\to 0}\frac{f(t,t)-f(0,0)}{t\,\|(1,1)\|}=$$

$$=\frac{1}{\sqrt{2}}\lim_{t\to 0}\frac{\sqrt{t^2}}{t}=\frac{1}{\sqrt{2}}\lim_{t\to 0}\frac{|t|}{t}$$

The limit does not exist, so the directional derivative does not exist. for $v=(1,1)$

Also, $\frac{\partial f(0,0)}{\partial x}=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=0$

Remark

We observe that $\boxed{\text{the existence of all directional derivatives does not guarantee continuity}}$

In 1D derivatives $\Rightarrow$ continuity.

In several dimensions

$\boxed{\text{all directional derivatives} \not\Rightarrow \text{continuity}}$

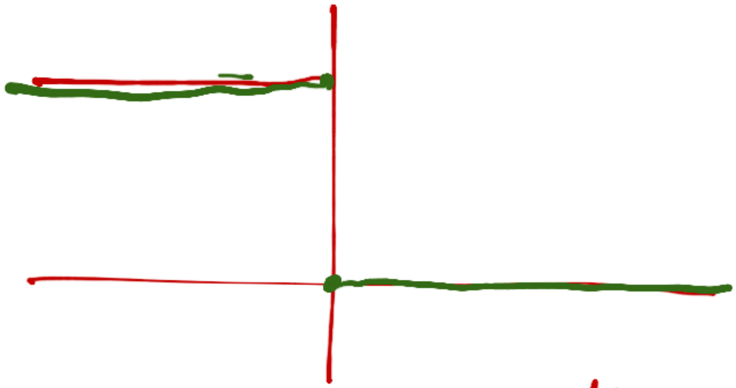
(we are actually approaching following lines)

Example:

$\begin{cases} 1 & \text{if } x=0 \text{ or } y=0 \end{cases}$ $\quad \frac{\partial f(0,0)}{\partial x} = 0 = \frac{\partial f(0,0)}{\partial y}$

$f(x,y) = $

$(x,y) \to (0,0)$



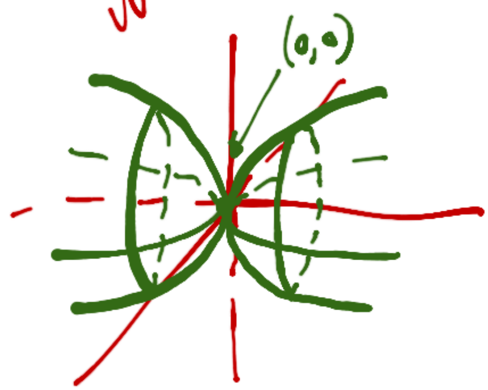
Example: $f(x,y) = x^{1/3} y^{1/3}$

$$\frac{\partial f(0,0)}{\partial x} = \frac{\partial f(0,0)}{\partial y} = 0$$

We must apply the definition since the $x^{1/3}$ and $y^{1/3}$ are not diff.

$$\frac{\partial f(x,y)}{\partial x} = \frac{1}{3} x^{-2/3} y^{1/3}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{1}{3} x^{1/3} y^{-2/3}$$



It looks necessary to have some kind of condition for the derivatives that implies, at least, continuity and also differentiability

diff. $\sim$ having a tangent plane.

$\ast$ (existence of derivatives in N-dimensions)